Mathematics Assessment Project

Formative Assessment Lesson Materials

Optimization Problems: *Boomerangs*

MARS Shell Center
University of Nottingham & UC Berkeley
Beta Version

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Optimization Problems: *Boomerangs*

Mathematical goals

This lesson unit is intended to help you assess how well students are able to:

- Interpret a situation and represent the constraints and variables mathematically.
- Select appropriate mathematical methods to use.
- Explore the effects of systematically varying the constraints.
- Interpret and evaluate the data generated and identify the optimum case, checking it for confirmation.
- Communicate their reasoning clearly.

Common Core State Standards

This lesson involves a range of *mathematical practices* from the standards, with emphasis on:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.

This lesson also asks students to select and apply mathematical content from across the grades, including the *content standards:*

A-CED: Create equations that describe numbers or relationships.
A-REI: Solve equations and inequalities in one variable.
Solve systems of equations.

Introduction

This lesson is designed to help students develop strategies for solving optimization problems. Such problems typically involve scenarios where limited resources must be used to greatest effect, as in, for example, the allocation of time and materials to maximize profit.

- Before the lesson, students attempt the problem individually. You then review their work and formulate questions for students to answer in order to improve their solutions.
- At the start of the lesson, students work alone answering your questions about the same problem.
- Students are then grouped, and engage in a collaborative discussion of the same task.
- In the same small groups, students are given sample solutions to comment on and evaluate.
- In a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- Finally, students revise their individual solutions, and comment on what they have learned.

Materials required

- Each individual student will need a calculator, and a copy of the assessment task *Boomerangs.*
- Each small group of students will need copies of the *Sample Responses to Discuss.*
- Graph paper should be kept in reserve, and used only when necessary or requested.
- There are also some projector resources to help you with whole-class discussions.

Time needed

Approximately fifteen minutes before the lesson, a one-hour lesson, and ten minutes in the next lesson (or for homework). Timings given are only approximate. Exact timings will depend on the needs of the class.

Before the lesson

Assessment task: *Boomerangs* (15 minutes)

Have the students do this task in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work, and to find out the kinds of difficulties students have with it. Then you will be able to target your help more effectively in the follow-up lesson.

Give out the task *Boomerangs*.

Introduce the task briefly, and help the class to understand the problem and its context. You could show examples of boomerangs.

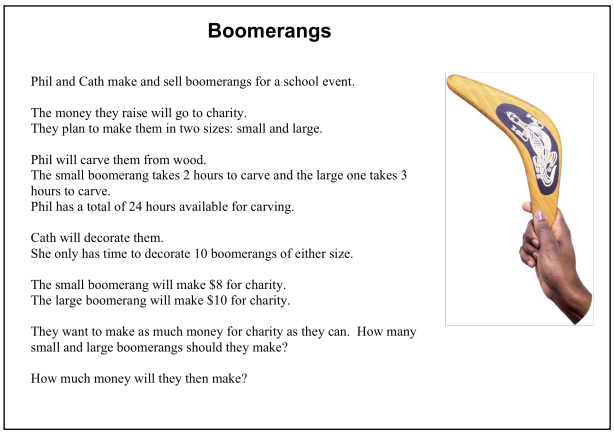
Boomerangs come from Australia, where they are used as weapons or for sport.

When thrown, they travel in a roughly elliptical path and return to the thrower.

Boomerangs are made in many different sizes.

Read through the questions and try to answer them as carefully as you can. Show all your work so that I can understand your reasoning.

As well as trying to solve the problem, I want you to see if you can present your work in an organized and clear manner.



Boomerangs

Phil and Cath make and sell boomerangs for a school event.

The money they raise will go to charity.
They plan to make them in two sizes: small and large.

Phil will carve them from wood.
The small boomerang takes 2 hours to carve and the large one takes 3 hours to carve.
Phil has a total of 24 hours available for carving.

Cath will decorate them.
She only has time to decorate 10 boomerangs of either size.

The small boomerang will make $8 for charity.
The large boomerang will make $10 for charity.

They want to make as much money for charity as they can. How many small and large boomerangs should they make?

How much money will they then make?

It is important that students are allowed to answer the questions without assistance, as far as possible.

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their different problem solving approaches. The purpose of doing this is to forewarn you of issues that will arise during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest that you write a list of your own questions, based on your own students' work, using the ideas below. You may choose to write questions on each student's work. If you do not have time to do this, just select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson. If students have used graphs or simultaneous equations in their solutions, add the relevant questions to their work. You may also want to note students with a particular issue, so that you can ask them about their difficulties in the formative lesson.

Common issues:	Suggested questions and prompts:
Student has difficulty getting started	• *What do you know?* • *What do you need to find out?*
Student makes an incorrect interpretation of the constraints and variables For example: The student has applied just one constraint, such as "Phil has only 24 hrs to make the boomerangs" or "Cath can only make 10 boomerangs." Or: The student has calculated the profit for making just one type of boomerang.	• *What figures in the task are fixed?* • *What can you vary?* • *What is the greatest number of small/large boomerangs they can make?* • *Have you used any unnecessary restrictions on the number of small and large boomerangs to be made?* • *Why can't they make 50 boomerangs?*
Student works unsystematically For example: The student shows three or four seemingly unconnected combinations, such as 5 small and 5 large boomerangs, then 10 large.	• *Can you organize the numbers of large and small boomerangs made in a systematic way?* • *What would be sensible values to try? Why?* • *How can you check that you remember all the constraints?* • *Do you cover all possible combinations? If not, why not?* • *How do you know for sure your answer is the best option?* • *Can you organize your work in a table?*
Student presents work poorly For example: The student presents the work as a series of unexplained numbers and/or calculations, or as a table without headings. Or: The student circles numbers, and it is left to the reader to work out why this is the answer as opposed to any other combination.	• *Would someone unfamiliar with your type of solution easily understand your work?* • *Have you explained how you arrived at your answer?*
Student has technical difficulties when using graphs For example: Lines are plotted inaccurately, axes are not labeled or the purpose of the graph is not explained. **Student has technical difficulties when using simultaneous equations** For example: A mistake is made when solving two correct simultaneous equations, or the correct solutions are obtained but the profit is not calculated.	• *Would someone unfamiliar with your type of solution easily understand your work?* • *How can you check your answer?* • *How do your answers help you solve the problem?*
Student produces a correct solution Student needs an extension task.	• *Can you now use a different method? For example, a table or graph, or algebra?* • *Is this method better than your original one? Why?* • *If the problem investigated how many boomerangs can be made in a month rather than 24 hours, would any method(s) be preferable to others?*

Suggested lesson outline

Improve individual solutions to the assessment task (10 minutes)

Return the assessment task papers to the students, and hand out calculators.

If you have not added questions to individual pieces of work, then write your list of questions on the board (excluding the ones for graphs and simultaneous equations). Students are to select questions appropriate to their own work, and spend a few minutes answering them.

Recall what we were looking at in a previous lesson. What was the task?
I have read your solutions, and I have some questions about your work.
I would like you to work on your own to answer my questions for about ten minutes.

Collaborative small-group work (10 minutes)

Organize the class into small groups of two or three students, and give out a fresh piece of paper to each group. Ask students to try the task again, this time combining their ideas.

Put your own work aside until later in the lesson. I want you to work in groups now.
Your task is to produce a solution that is better than your individual solutions.

While students work in small groups you have two tasks, to note different student approaches to the task, and to support student problem solving.

Note different student approaches to the task

You can then use this information to focus a whole-class discussion towards the end of the lesson. In particular, note any common mistakes. For example, are students consistently using all the constraints, or are they imposing unnecessary constraints? Also note whether students are using algebra and, if so, how they are using it.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions that help students to clarify their thinking. You may discover that some students experience some difficulty in keeping more than one constraint at a time in mind. In that case, you may ask them to consider these three questions:

If they were to make only small boomerangs, how much money would they make?
If they were to make two small boomerangs, how many large ones could they also make?
How much money would they make?

For the first question, Cath's time is the limiting constraint, whereas in the second question, Phil's time is more significant. Students who organize their work into a table may choose to use column headings for "Time needed for Phil" and "Time needed for Cath," which they can use to check that both constraints have been met.

To help students really struggling with the task, use the questions on the previous page to support your own questioning. In particular, if students find it difficult to get started, these questions may be useful:

Try some examples. What happens if they make three small and one large boomerang?
What would be sensible values to try? Why?
Can you organize the numbers of large and small boomerangs made in a systematic way?

If the whole class is struggling on the same issue, write relevant questions on the board. You could also ask students who performed well on the assessment to help struggling students. If students are having difficulty making any progress at all, you could hand out two pieces of sample work to model problem solving methods.

Collaborative analysis of *Sample Responses to Discuss* (20 minutes)

After students have had sufficient time to attempt the problem, give each group of students a copy of each of the four *Sample Responses to Discuss,* and ask for written comments. This task gives students the opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy.

Imagine you are the teacher and have to assess this work.
Correct the work, and write comments on the accuracy and organization of each response.

Each of the sample responses poses specific questions for students to answer. In addition to these, you could ask students to evaluate and compare responses. To help them do more than check to see if the answer is correct, you may wish to use the projector resource *Evaluating Sample Responses to Discuss*:

- *What do you like about the work?*
- *How has each student organized the work?*
- *What mistakes have been made?*
- *What isn't clear?*
- *What questions would you like to ask this student?*
- *In what ways might the work be improved?*

You may decide there is not enough time for each group to work through all four pieces of work. In that case, be selective about what you hand out. For example, groups that have successfully completed the task using one method will benefit from looking at different approaches. Other groups that have struggled with a particular approach may benefit from seeing a student version of the same strategy.

During the small-group work, support the students as before. Note similarities and differences between the sample approaches, and those approaches students took in the small-group work. Also check to see which methods students have difficulties in understanding. This information can help you focus the next activity, a whole-class discussion.

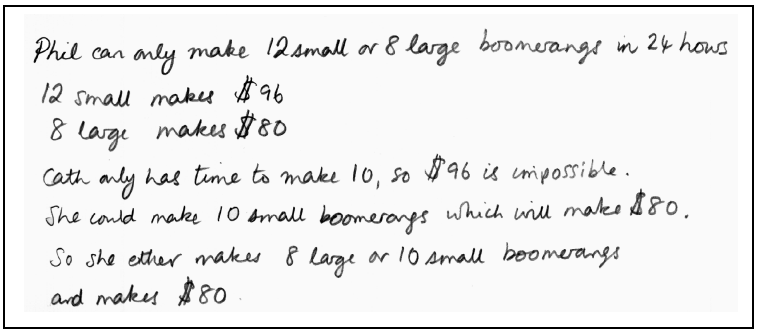
Plenary whole-class discussion: comparing different approaches (10 minutes)

Organize a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on those parts of the small-group tasks that students found difficult. Ask the students to compare the different solution methods.

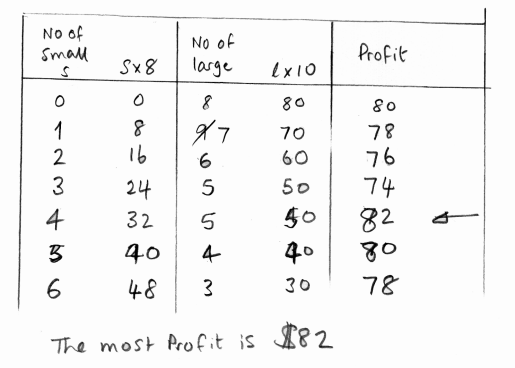
Which approach did you like best? Why?
Which approach did you find most difficult to understand?

To critique the different strategies use the questions on the slide *Evaluating Sample Responses to Discuss* and the worksheets *Sample Responses to Discuss*.

Alex has realized that you have to take account of both constraints: Phil's time for making the boomerangs and Cath's time for decorating them. Alex has not examined different combinations of cases.


Phil can only make 12 small or 8 large boomerangs in 24 hours
12 small makes $96
8 large makes $80
Cath only has time to make 10, so $96 is impossible.
She could make 10 small boomerangs which will make $80.
So she either makes 8 large or 10 small boomerangs
and makes $80

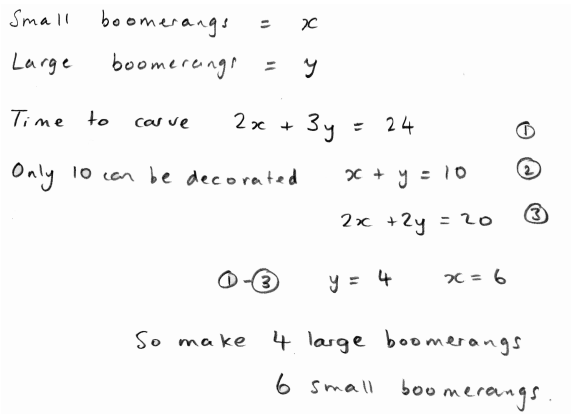
Danny has found an effective way to organize his work, using a table. He has made some mistakes in this table, however. Part of the problem is that he loses track of the two constraints. It might have been helpful for him to include two additional columns headed: "Time needed (≤ 24 hours)" and "Total number made (≤ 10)." Then he could test each case and put a check mark if it satisfies both constraints.



No of Small s	8s	No of large l	10l	Profit
0	0	8	80	80
1	8	~~9~~ 7	70	78
2	16	6	60	76
3	24	5	50	74
4	32	5	50	82 ←
5	40	4	40	80
6	48	3	30	78

The most Profit is $82

Jeremiah has tried an algebraic approach and has hit upon the correct solution. However, he has used equalities rather than inequalities. He needs to calculate the total profit to complete the question.



Small boomerangs = x

Large boomerangs = y

Time to carve $2x + 3y = 24$ ①

Only 10 can be decorated $x + y = 10$ ②

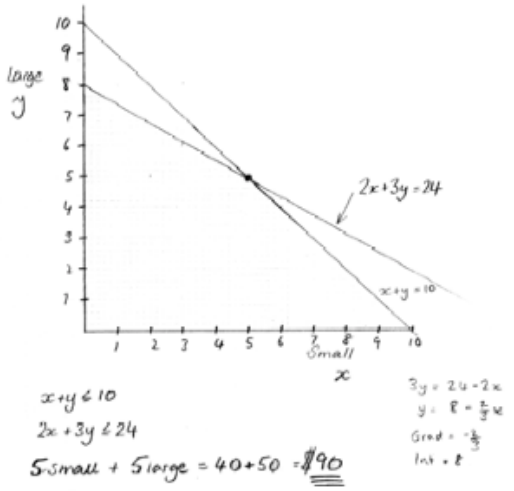
$2x + 2y = 20$ ③

① − ③ $y = 4$ $x = 6$

So make 4 large boomerangs

6 small boomerangs

Tanya has used a graphical approach, but her graph of $2x + 3y = 24$ is inaccurate and should be redrawn. This graph is powerful in that it shows the entire feasible solution space—the integer points on the grid. She has not explained why her method will give the greatest profit.



Review individual solutions to the assessment task (10 minutes)

Ask students to read through their original responses to the task.

Read through your original solution and think about your work this lesson.

Write down what you have learned during the lesson.

Which method would you prefer to use if you were doing the task again? Why?

Encourage students to compare the new approaches they met during the lesson with their original method.

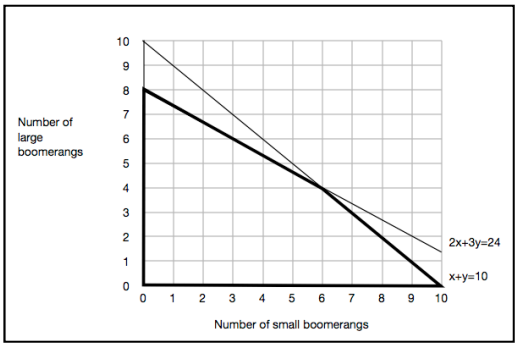
Some teachers set this task as homework.

Solutions

If one assumes that ten boomerangs are made, then the following table of possibilities may be made.

The constraint on carving hours is broken when more than four large boomerangs are made.

Number of small	Number of large	Total number (≤10)	Carving hours (≤ 24)	Profit made
10	0	10	20	80
9	1	10	21	82
8	2	10	22	84
7	3	10	23	86
6	4	10	24	88
5	5	10	25	90



This approach, however, does not include the possibility of making fewer than ten boomerangs. A more complete approach would be to draw a graph showing all possibilities.

The possible combinations to be checked are the integer points within the bold region on the graph.
The maximum profit occurs, however, when six small and four large boomerangs are made.
This profit is $88.

(This can be seen graphically by drawing lines of constant profit on the graph, e.g. $8x + 10y = 80$. This idea may emerge in discussion.)

Boomerangs

Phil and Cath make and sell boomerangs for a school event.
The money they raise will go to charity.

They plan to make them in two sizes: small and large.

Phil will carve them from wood.
The small boomerang takes 2 hours to carve and the large one takes 3 hours to carve.
Phil has a total of 24 hours available for carving.

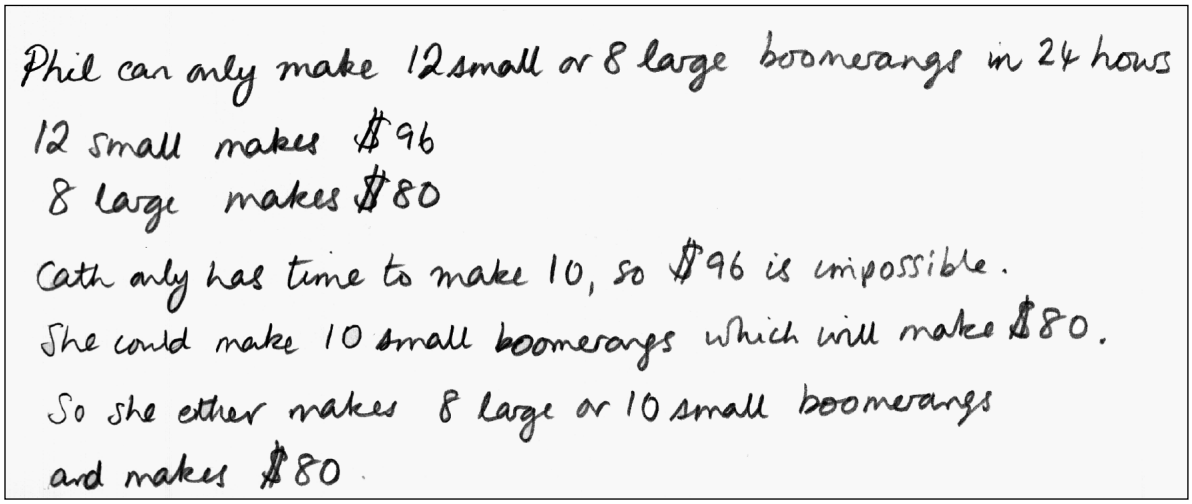
Cath will decorate them.
She only has time to decorate 10 boomerangs of either size.

The small boomerang will make $8 for charity.
The large boomerang will make $10 for charity.
They want to make as much money for charity as they can.

How many small and large boomerangs should they make?

How much money will they then make?

Sample Responses to Discuss: Alex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So she either makes 8 large or 10 small boomerangs
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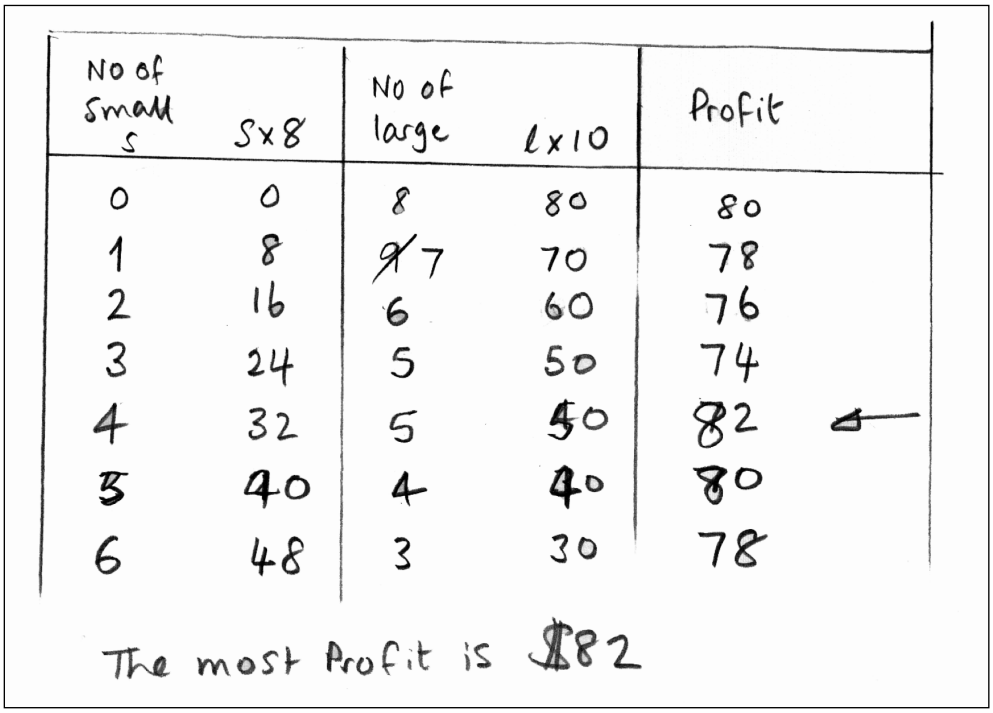
What assumptions has Alex made?

Are these assumptions correct? Explain your answer.

General comments:

Sample Responses to Discuss: Danny



No of smalls	S x 8	No of large	L x 10	Profit
0	0	8	80	80
1	8	~~9~~ 7	70	78
2	16	6	60	76
3	24	5	50	74
4	32	5	50	82 ←
5	40	4	40	80
6	48	3	30	78

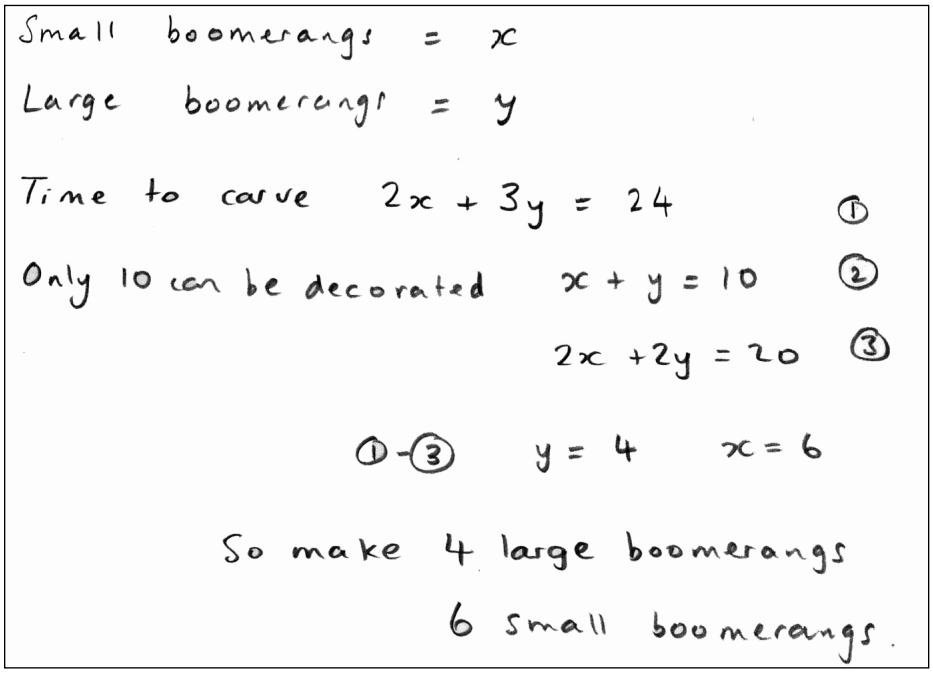
The most Profit is $82

Why do you think Danny starts with 0 small and 8 large boomerangs and stops at 6 small and 3 large boomerangs?

What piece of information has Danny forgotten to use?

General comments:

Sample Responses to Discuss: Jeremiah

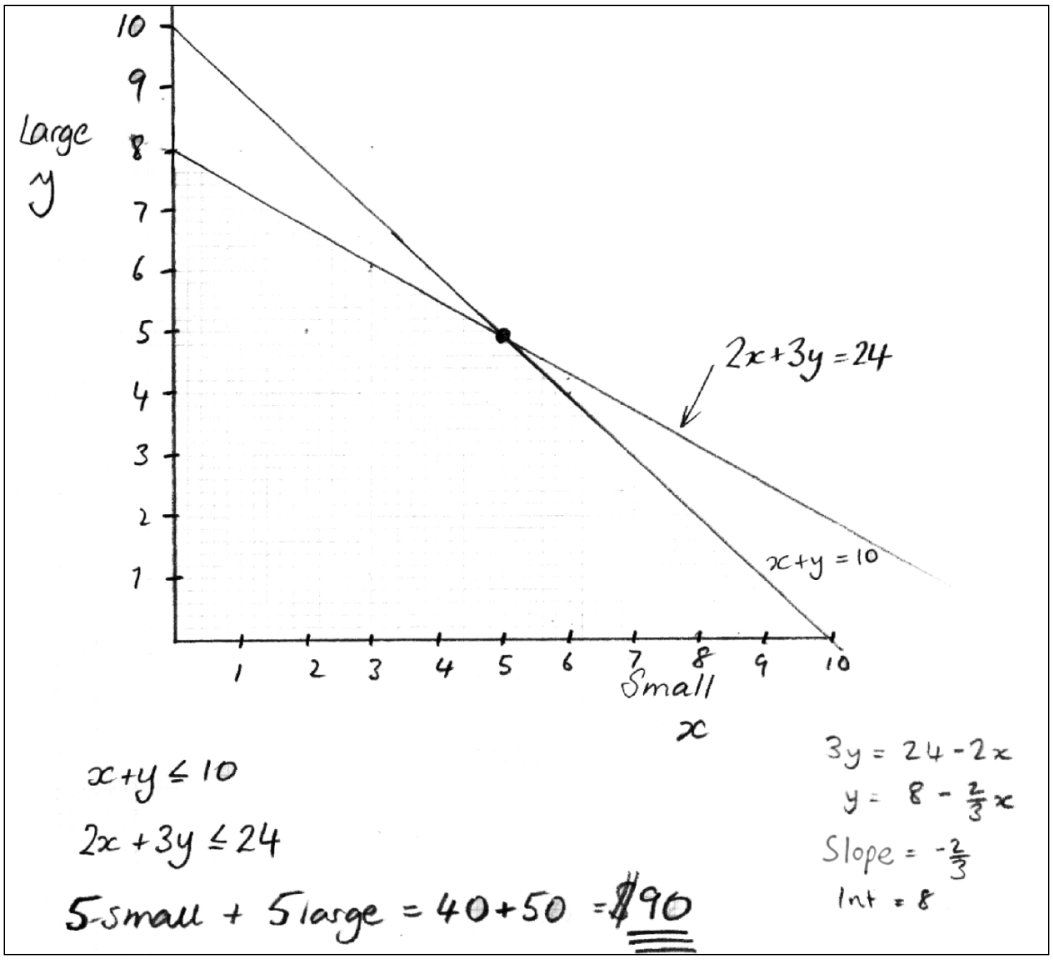


Is it correct to use the equals sign in equations 1, 2 and 3? Explain your answer?

Why is Jeremiah's solution incomplete?

General comments:

Sample Responses to Discuss: Tanya



What is the purpose of the graph?

What is the point of figuring out the slope and intercept?

General comments:

Evaluating *Sample Responses to Discuss*

- What do you like about the work?
- How has each student organized the work?
- What mistakes have been made?
- What isn't clear?
- What questions do you want to ask this student?
- In what ways might the work be improved?

Alex's solution

Phil can only make 12 small or 8 large boomerangs in 24 hours

12 small makes $96

8 large makes $80

Cath only has time to make 10, so $96 is impossible.

She could make 10 small boomerangs which will make $80.

So she either makes 8 large or 10 small boomerangs

and makes $80.

Danny's solution

No of smalls	8x	No of large	10x	Profit	
0	0	8	80	80	
1	8	~~9~~ 7	70	78	
2	16	6	60	76	
3	24	5	50	74	
4	32	5	~~5~~0	~~8~~2	←
~~5~~	~~40~~	4	~~4~~0	~~8~~0	
6	48	3	30	78	

The most Profit is ~~$~~82

Jeremiah's solution

Small boomerangs = x

Large boomerangs = y

Time to carve $2x + 3y = 24$ ①

Only 10 can be decorated $x + y = 10$ ②

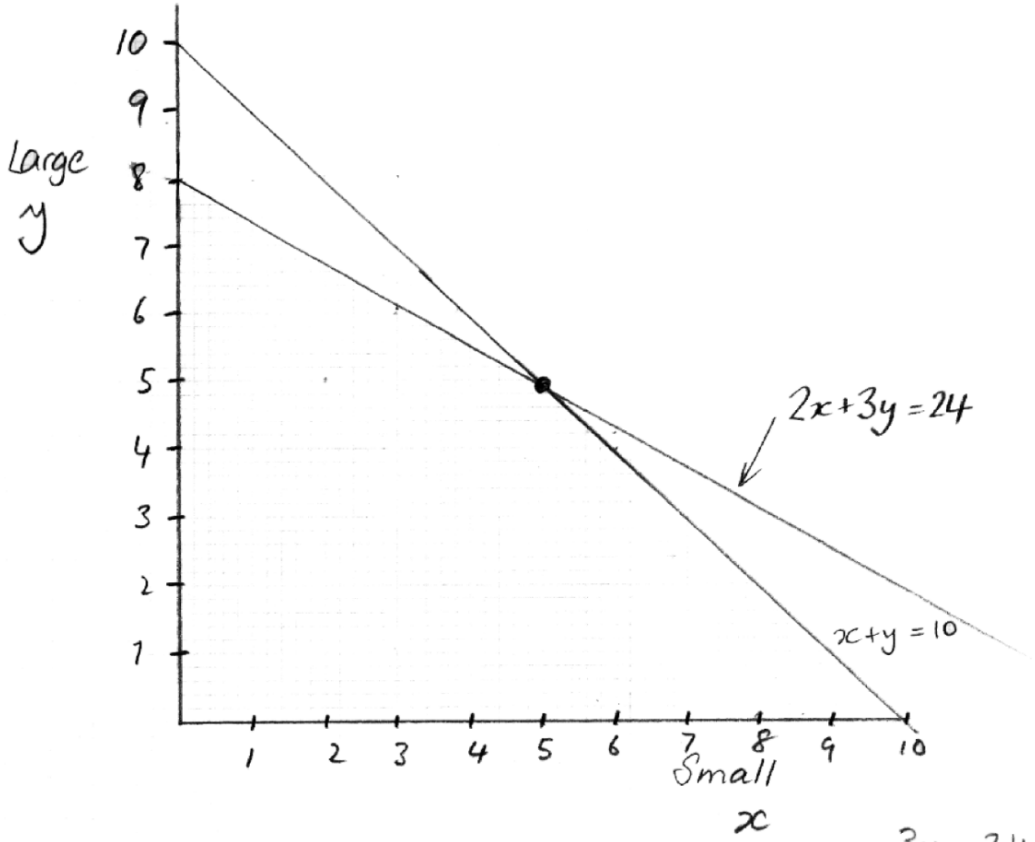
$2x + 2y = 20$ ③

① − ③ $y = 4$ $x = 6$

So make 4 large boomerangs

6 small boomerangs.

Tanya's solution



$x + y \leq 10$

$2x + 3y \leq 24$

5 small + 5 large = 40 + 50 = £90

$3y = 24 - 2x$

$y = 8 - \frac{2}{3}x$

Slope $= -\frac{2}{3}$

Int = 8